More examples of finding Taylor series.

Name: Taylor series at $x=0$

$$T_\infty(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Also known as McLauren series ($x=0$ Taylor expansion).

$\boxed{\text{Examples}}$

Find the Taylor series of $f(x) = \frac{1}{x}$ @ $x=-2$.

$$f(x) = x^{-1} \overset{x=-2}{=} (-2)^{-1} = -\frac{1}{2}$$

$$f'(x) = (-1)x^{-2} \overset{x=-2}{=} (-1)(-2)^{-2} = -\frac{1}{4} = -\frac{1}{2^2} = -\frac{1!}{2^2}$$

$$f''(x) = (-1)(-2)x^{-3} \overset{x=-2}{=} (-1)(-2)(-2)^{-3} = \frac{-1}{4} = \frac{-1}{2^2} = \frac{-2!}{2^3}$$

$$f'''(x) = (-1)(-2)(-3)x^{-4} \overset{x=-2}{=} (-1)(-2)(-3)(-2)^{-4} = \frac{-3!}{2^4}$$

$$f^{(n)}(x) = \frac{-n!}{2^{n+1}}$$

Taylor series @ $x=-2$

$$T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!}(x+2)^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{-\cancel{k!}}{2^{k+1}} \frac{1}{\cancel{k!}}\right)(x+2)^k$$

$$\boxed{= \sum_{k=0}^{\infty} -\frac{(x+2)^k}{2^{k+1}}}$$

$$\boxed{= -\frac{1}{2} - \frac{1}{2^2}(x+2) - \frac{1}{2^3}(x+2)^2 - \frac{1}{2^4}(x+2)^3 - \dots}$$

Alternate way to get the last answer:

$f(x) = \frac{1}{x}$ near $x = -2$

$$f(x) = \frac{1}{(x+2)-2} = -\frac{1}{2-(x+2)} = -\frac{1}{2}\left(\frac{1}{1-\left(\frac{x+2}{2}\right)}\right)$$

$$= \frac{-\frac{1}{2}}{1-\left(\frac{x+2}{2}\right)}$$

← geometric series formula

$a = -\frac{1}{2}$

$r = \frac{x+2}{2}$.

geometric

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left(\frac{x+2}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{(x+2)^n}{2\cdot 2^n}$$

$$f(x) = \boxed{\sum_{n=0}^{\infty} -\frac{(x+2)^n}{2^{n+1}}}$$

same as calculated above.

This equals the function

(the series converges to the function) as long as $|r| < 1 \iff \left|\frac{x+2}{2}\right| < 1 \iff |x+2| < 2$

$\iff -2 < x+2 < 2$

$\iff -4 < x < 0$

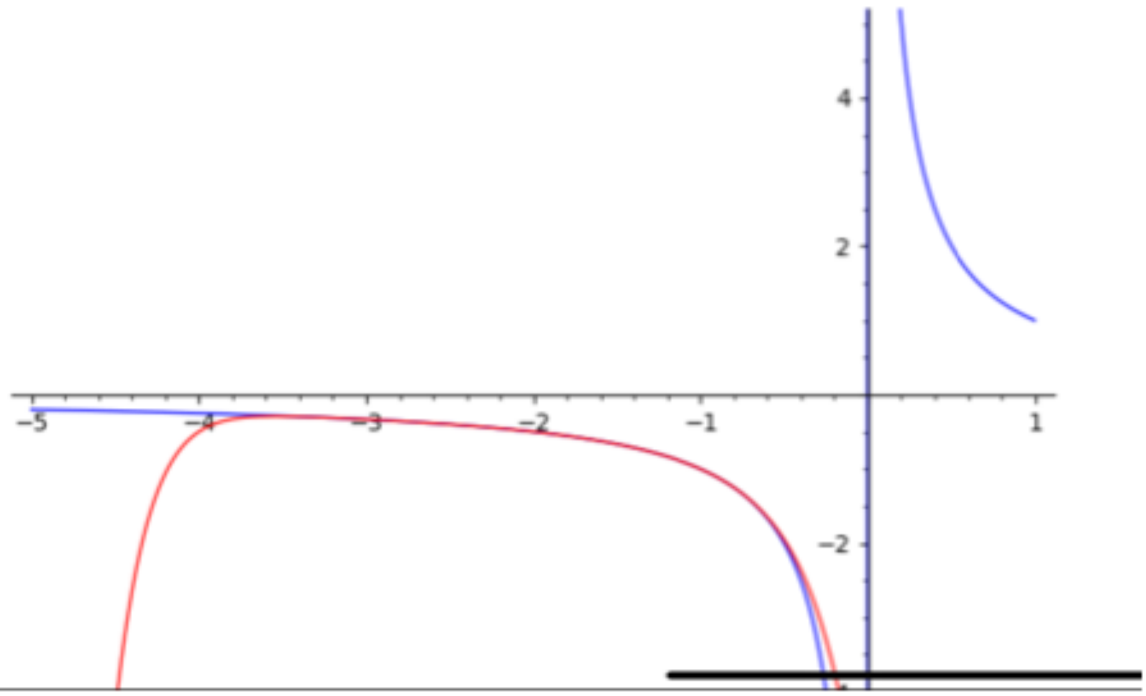
So the Taylor series for $\frac{1}{x}$ @ $x=-2$ converges to $\frac{1}{x}$ as long as $-4<x<0$.

```
f(x)=1/x
g(x)=taylor(f(x),x,-2,12)
show(f(x))
show(g(x))
a=plot(f(x),(x,-5,1),ymin=-5,ymax=5)
b=plot(g(x),(x,-5,1),ymin=-5,ymax=5,color="red")
show(a+b)
```

Evaluate

$$\frac{1}{x}$$

$$-\frac{1}{8192}(x+2)^{12}-\frac{1}{4096}(x+2)^{11}-\frac{1}{2048}(x+2)^{10}-\frac{1}{1024}(x+2)^{9}-\frac{1}{512}(x+$$



This tells us: we can use the geometric series to help us find Taylor series without taking general derivatives at a point.

$\boxed{\text{Example}}$ Let's compute the Taylor series of $f(x)=\arctan(x)$ @ $x=0$. Where does it converge to $f(x)$?

Taylor series:

$$f(x) = \arctan(x) \overset{x=0}{=} 0$$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} \overset{x=0}{=} 1$$

$$f''(x) = (-1)(1+x^2)^{-2}\cdot(2x) \overset{x=0}{=} 0$$

$$f'''(x) = \text{ouch}$$

Taylor polynomial of degree 2:

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$T_2(x) = 0 + x + 0x^2$$

$$\boxed{T_2(x) = x}$$

---

another approach:

$$(\arctan(x))' = \frac{1}{1+x^2}$$

$$= \frac{1}{1-(-x^2)}$$

geometric $a=1$

$r=-x^2$

$a + ar + ar^2 + ar^3$

$$(\arctan(x))' = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8$$

converges if $|r|<1 \iff |-x^2|<1$

$|x|^2<1$ $|x|<1$

$$\Rightarrow \arctan(x) + C = \int (\arctan(x))' \, dx$$

antiderivative

$$\arctan(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

converges if $|x| < 1$.
$-1 < x < 1$.

What's C?

plug in $x = 0$

$$\arctan(0) + C = 0 - 0 + 0 - \dots$$

$$0 + C = 0$$

$$\Rightarrow C = 0$$

$$\boxed{\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$$

converges for $-1 < x < 1$

$x = -1$

$$-1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} = -\infty$$

diverges

$x = 1$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \longrightarrow$ actually converges.

Some more fun Taylor polynomials & series.

Easy ones:

- $y = e^x$ @ $x = 0$.

$$f(x) = e^x \overset{x=0}{=} 1$$
$$f'(x) = e^x = 1$$
$$f''(x) = e^x = 1$$
$$f'''(x) = e^x = 1$$

$$T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- $y = \sin(x)$ @ $x = 0$

$$f(x) = \sin(x) \overset{x=0}{=} 0$$
$$f'(x) = \cos(x) = 1$$
$$f''(x) = -\sin(x) = 0$$
$$f'''(x) = -\cos(x) \overset{x=0}{=} -1$$
$$f''''(x) = \sin(x) = 0$$

$$T_\infty(x) = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 + \cdots$$

$$\boxed{T_\infty(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

• $y = \cos(x)$ @ $x = 0$

$f(x) = \cos(x) \overset{x=0}{=} 1$ $\quad T_\infty(x) = 1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \cdots$

$f'(x) = -\sin(x) = 0$

$f''(x) = -\cos(x) = -1$ $\quad \boxed{T_\infty(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$

$f'''(x) = \sin(x) = 0$

$f^{iv}(x) = \cos(x) = 1$

<u>Consider:</u> $e^{i\theta}$. Using Taylor series formula:

$i^2 = -1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$= 1 + i\theta + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \frac{-\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \frac{i\theta^9}{9!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Example:

$$e^{2i\theta} = \left(e^{i\theta}\right)^2 = e^{i(2\theta)}$$

$$\left(\cos\theta + i\sin\theta\right)^2 = \cos(2\theta) + i\sin(2\theta)$$

$$\cos^2\theta - \sin^2\theta + 2\cos\theta\, i\sin\theta$$

$$\Rightarrow \left(\cos^2\theta - \sin^2\theta\right) + i\left(2\sin\theta\cos\theta\right)$$

$$= \cos(2\theta) + i\sin(2\theta)$$

Can find new Taylor series from old ones:

(Ex) Find the Taylor series of

$F(x) = x^2 e^{x^5}$ centered at $x=0$.

$F(0) = 0$

$F'(x) = 2xe^{x^5} + 5x^6 e^{x^5} = 0$

$\vdots$ ouch.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^5} = 1 + x^5 + \frac{x^{10}}{2!} + \frac{x^{15}}{3!} + \cdots$$

$$\boxed{x^2 e^{x^5} = x^2 + x^7 + \frac{x^{12}}{2!} + \frac{x^{17}}{3!} + \cdots}$$

$$e^{x^5} = \sum_{n=0}^{\infty} \frac{(x^5)^n}{n!}$$

$$x^2 e^{x^5} = \sum_{n=0}^{\infty} \frac{x^2 (x^5)^n}{n!}$$

$$\boxed{x^2 e^{x^5} = \sum_{n=0}^{\infty} \frac{x^{5n+2}}{n!}}$$

☆ How do we know if a Taylor series converges to the function, and where?

(We know the answer for $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ converges to the function $\iff |x| < 1$.

How do we tell for any function?

The Taylor-Lagrange Remainder formula tells us how accurate a Taylor polynomial is.

Examples

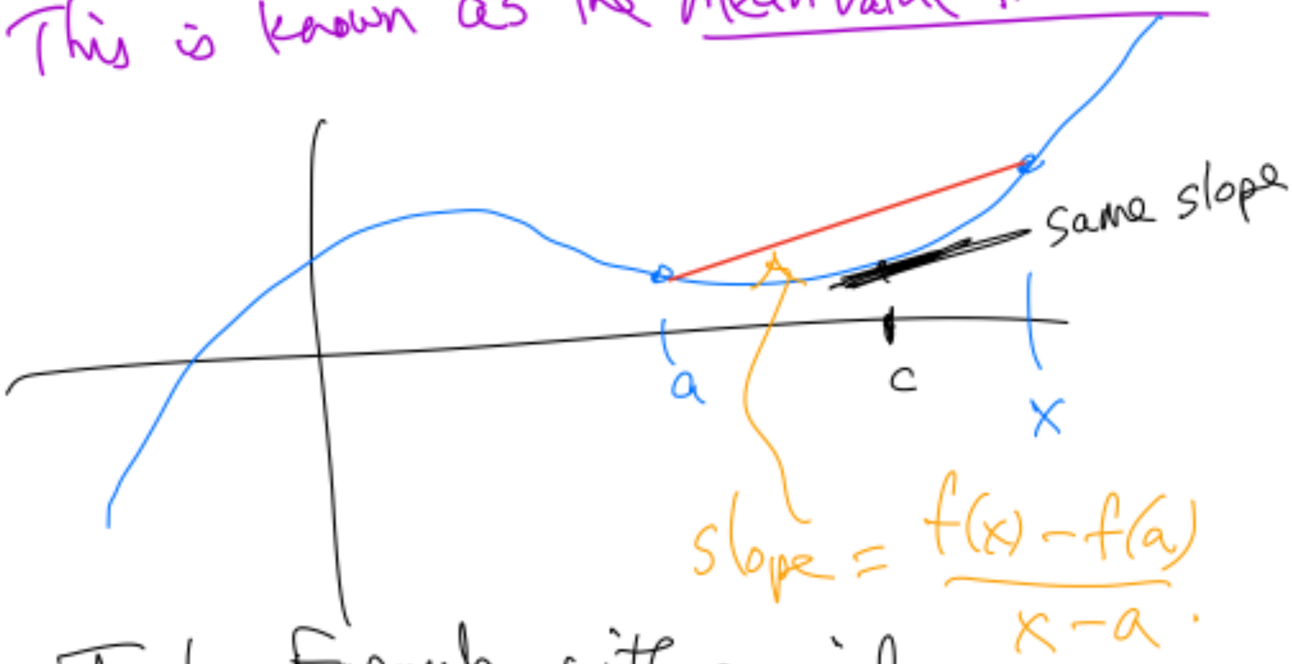
① $0^{\underline{th}}$ degree Taylor polynomial:

$$f(x) \approx f(a)$$

$$f(x) - f(a) = f'(c)\,(x-a)$$

i.e. $$f'(c) = \frac{f(x)-f(a)}{x-a}$$

where c is a number between a & x.
This is known as the Mean Value Theorem.



Taylor Formula with remainder for
$f(x) \approx f(a)$ $0^{\underline{th}}$ degree Taylor polynomial

is $$f(x) = f(a) + f'(c)\,(x-a)$$

(c is a point between a & x.)

exact equals sign (as long as f is differentiable).

More general Taylor polynomial

$$f(x) \approx T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor formula with Lagrange Remainder:

$$f(x) = T_n(x) + \overset{\text{remainder}}{R_n(x)}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x).}$$

where

c is between a & x.

There exists at least one value of c to make this true.

A function is = to its Taylor series $T_\infty(x)$ ~~if~~

$$\lim_{n\to\infty} R_n(x) = 0.$$

<u>Example</u>: for which x does the Taylor series of e^x converge to e^x?

Taylor Lagrange formula

$$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}}_{R_n(x).}$$

$$f(x) = e^x$$

$$f^{(n+1)}(x) = e^x \Rightarrow f^{(n+1)}(c) = e^c$$

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$$ where c is between 0 & x.

Taylor series converges at x

$\Longleftrightarrow \lim_{n\to\infty} R_n(x) = 0$

$\Longleftrightarrow \lim_{n\to\infty} \dfrac{e^c x^{n+1}}{(n+1)!}$

c between 0 & x.

$$0 \le \left|\frac{e^c x^{n+1}}{(n+1)!}\right| \le \frac{e^{|x|}|x|^{n+1}}{(n+1)!}$$

if $n \to \infty$, is this going to 0?

Both numerator & denominator are going to ∞ ← hard to tell.

$|x|$ is fixed, and $n \to \infty$.

eg $|x| = 7$ $\quad \dfrac{e^7\, 7^{n+1}}{(n+1)!}$, as $n \to \infty$

$n=1$ $\dfrac{e^7\cdot 7^2}{1\cdot 2}$, $n=2$ $\dfrac{e^7\, 7^3}{1\cdot 2\cdot 3}$, $\dfrac{e^7\, 7^4}{1\cdot 2\cdot 3\cdot 4}$,

$\ldots$, $\dfrac{e^7\, 7^7}{7!}$, $\dfrac{e^7\, 7^8}{7!\cdot 8}$,

$\dfrac{e^7\, 7^7\cdot 7\cdot 7}{7!\cdot 8\cdot 9}$, $\dfrac{e^7\cdot 7^7\cdot 7\cdot 7\cdot 7}{7!\; 8\cdot 9\cdot 10}$,

eventually the denominator wins the contest, and the

$$\lim_{n\to\infty} R_n(x) = 0.$$

This example was for $|x|=7$, but the same thing happens for any value of $|x|$. Eventually,

$$(n+1)! >>> |x|^{n+1}.$$

Thus for all $x \in \mathbb{R}$

$$\boxed{e^x = T_\infty(x)}\ .$$

Similarly, the Taylor series for $\sin(x)$ & $\cos(x)$ converge to the function for all values of x.